

Dispersion effects on buoyancy-driven convection  
in stratified flows through porous media

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Abstract

The effect of hydrodynamic dispersion on the onset of convection in flows through porous media is studied theoretically. The medium is isotropic, and bounded by two horizontal impermeable planes having a constant concentration difference. Pressure-driven as well as thermally-driven basic flows are considered. The investigations are valid in the limit of small and large Peclet numbers. The analysis shows that the onset of convection is independent of the longitudinal dispersion coefficient, while lateral dispersion always has a stabilizing effect. The preferred mode of disturbance is stationary, being rolls with axes aligned in the direction of the basic flow (longitudinal rolls).

## INTRODUCTION

Buoyancy-driven convection in porous media is important from a geophysical point of view. Thus, this phenomenon may affect groundwater motions in areas with geothermal activity. [Wooding, 1957]. In some situations solute as well as temperature gradients will be present, as for example in connection with hot salty springs, or fertilizer migration in saturated heated soil. Concerning heat and mass transfer in porous media, it is well known that the effect of dispersion may be significant. Introducing the concept of average field variables, the heat and mass dispersion effects can be explained as resulting from the convective transport terms  $\overline{\vec{v}'T'}$  and  $\overline{\vec{v}'S'}$ , where  $\vec{v}'$ ,  $T'$  and  $S'$  are the deviations from the mean velocity, the mean temperature and the mean concentration in the neighbourhood of a point [Rumer, 1972]. The bars denote average values defined over a macroscopic control volume being large compared to the characteristic grain size, but small compared to the characteristic dimensions of the porous medium as a whole.

The importance of dispersion is known to be an increasing function of the Peclet number, defined with respect to the characteristic grain diameter. In the present paper we consider Darcian flows, which implies small Reynolds numbers. Since the Reynolds and Peclet numbers are of about the same order of magnitude for thermal convection in a water-saturated medium, which is the most interesting problem in this connection, heat dispersion effects will be entirely neglected.

For a flow involving a solute, however, the situation may be different. Since the mass diffusivity is much smaller than the

heat diffusivity, the ratio being about 1/100 for salt and heat in water, dispersion effects may be significant even in the Darcian flow regime; see also Dagan [1972]. This has motivated the present study, in which we investigate the influence of on buoyancy-driven convection in a hydrodynamic dispersion solution in a porous medium. For small velocities (or Peclet numbers), it is known that the coefficient of dispersion is proportional to the square of the velocity [Poreh, 1965]. Accordingly dispersion will not influence the onset of convection in a porous layer where the basic state is motionless. When the basic state involves a fluid flow of finite velocity, however, dispersion will contribute linearly to the stability equations, and hence affect the breakdown of stability [Rubin, 1974].

In the present paper the stability of two different types of basic flows have been studied. The first one is pressure-driven and isothermal, while the second is a thermally-driven shear flow. The analysis is valid in the limit of small and large Peclet numbers in the first case, while in the second case only small Peclet numbers have been considered.

### GOVERNING EQUATIONS

Consider three-dimensional convection in a porous medium bounded by two horizontal impermeable planes, which are taken to perfect conductors of heat and concentration. The planes are separated by a distance  $h$ , which is assumed to be small compared to the characteristic horizontal dimensions. The concentration difference between upper and lower plane is taken to be  $\Delta S$ , and the heavier fluid is at the top. We choose the frame of reference such that the  $x_*$ - and  $y_*$ - axes are situated in the middle of the layer,

while the  $z_*$ -axis is directed upwards. The respective unit vectors are denoted by  $(\vec{i}, \vec{j}, \vec{k})$ . Dimensionless (unstarred) quantities may conveniently be introduced by taking

$$h, h^2/\kappa_{mS}, \kappa_{mS}/h, \rho_0 v \kappa_{mS}/K, \Delta S, v \kappa_{mS}/Kg\gamma_1 h \quad (1)$$

as units of length, time,  $t$ , velocity,  $\vec{v} = (u, v, w)$ , pressure,  $p$ , concentration,  $S$ , and temperature,  $T$ . Utilizing the Boussinesq approximation, and assuming that the density is linear in temperature and concentration, the governing equations may be written in dimensionless form

$$\nabla p = - \vec{v} - Ra S \vec{k} + T \vec{k} \quad (2)$$

$$\nabla \cdot \vec{v} = 0 \quad (3)$$

$$\frac{\partial S}{\partial t} + \vec{v} \cdot \nabla S = \nabla \cdot \mathcal{D} \cdot \nabla S \quad (4)$$

$$\tau (c \frac{\partial T}{\partial t} + \vec{v} \cdot \nabla T) = \nabla^2 T \quad (5)$$

where  $Ra = Kg\gamma_2 \Delta Sh / v \kappa_{mS}$  is the solute Rayleigh number, and  $\mathcal{D}$  is the tensor of hydrodynamic dispersion. In (1)-(5)  $\kappa_{mS}$  and  $\kappa_{mT}$  are the effective diffusivities in the porous medium of the dissolved substance and the temperature, respectively,  $v$  the kinematic viscosity,  $K$  the permeability,  $\rho$  the density,  $\rho_0$  a standard density and  $g$  the acceleration of gravity. The "expansion coefficients"  $\gamma_1$  and  $\gamma_2$  are given by  $\gamma_1 = -(1/\rho)(\partial \rho / \partial T_*)_{p_*, S_*}$ ,  $\gamma_2 = (1/\rho)(\partial \rho / \partial S_*)_{p_*, T_*}$ . Finally,  $\tau = \kappa_{mS} / \kappa_{mT}$  and  $c = (\rho c_p)_f / (\rho c_p)_m$ , where  $c_p$  is the specific heat at constant pressure and the subscripts  $f$  and  $m$  denote fluid and fluid saturated medium, respectively.

For a derivation of the heat equation (5), see Katto and Masuoka [1967].

The general form of the dispersion tensor  $\mathcal{D}$  has been investigated by several authors; see for example the review by Bear [1969]. In the present study we restrict ourselves to isotropic homogeneous media, and the effect of hydrodynamic dispersion will be considered only for the two limiting cases of small and large Peclet numbers.

When  $Pe$  is small, diffusion dominates. According to Poreh [1965], we then may write

$$D_{ij} = (1 + \epsilon_2 |\vec{V}|^2) \delta_{ij} + (\epsilon_1 - \epsilon_2) u_i u_j \quad (6)$$

where non-dimensional terms have been introduced according to (1). In (6)  $\delta_{ij}$  is the Kronecker delta,  $u_i$  the velocity components and  $\epsilon_1, \epsilon_2$  are constants depending on the geometry of the medium. In a reference system where one of the coordinate axis coincides with the flow direction,  $\epsilon_1$  and  $\epsilon_2$  are the longitudinal and the lateral dispersion coefficients, respectively. Their values have been obtained analytically by Saffman [1960], from which we find

$$\frac{\epsilon_1}{\epsilon_2} = \frac{8}{3} \quad (7)$$

Further  $\epsilon_1, \epsilon_2$  are small, being of the order  $(d/h)^2$ , where  $d$  is the characteristic grain diameter.

When  $Pe$  is large, diffusion processes can be neglected compared to mechanical dispersion. In such cases we take

$$D_{ij} = \eta_2 |\vec{V}| \delta_{ij} + (\eta_1 - \eta_2) \frac{u_i u_j}{|\vec{V}|} \quad (8)$$

where  $\eta_1$  and  $\eta_2$  are constants. The relationship between  $\eta_1$  and  $\eta_2$  are somewhat uncertain. Bear [1969] obtains  $\eta_1/\eta_2 > 2$  and reports that values between 6 and 24 are given in the literature by various investigators. Concerning the magnitude of  $\eta_1, \eta_2$  in this case, experiments for one-dimensional flow [Pfankuch, 1963] give

$$\eta_1 = 1.8 \frac{d}{h} \quad (9)$$

while Dagan [1972] summarizes that  $\eta_1 \approx 1.4 \frac{d}{h}$  for  $Pe > 100$ .

#### THE STABILITY OF UNIFORM ISOTHERMAL FLOW

##### Results for small Peclet numbers.

For a given horizontal pressure gradient it is easily seen that the system of equations (2)-(5) permits a steady, isothermal solution of the form

$$\begin{aligned} u &= U = \text{constant}, \quad v = w = 0 \\ S(z) &= z \end{aligned} \quad (10)$$

Neglecting dispersion effects, Prats [1966] has analyzed the stability of this flow. Perturbating the solution (10), the velocity and concentration may be written

$$\begin{aligned} \vec{v} &= U\vec{i} + \hat{\vec{v}}(x, y, z, t) \\ S &= S(z) + \hat{s}(x, y, z, t) \end{aligned} \quad (11)$$

where the carets denote perturbation quantities. Since  $\vec{k} \cdot (\nabla \times \vec{v}) = 0$  and  $\nabla \cdot \vec{v} = 0$  from (2) and (3),  $\vec{v}$  is a poloidal vector. Hence the

perturbation velocity may be written

$$\hat{\vec{v}} = \nabla \times \nabla \times \vec{k} \hat{\psi} \quad (12)$$

or explicitly

$$\{\hat{u}, \hat{v}, \hat{w}\} = \{\hat{\psi}_{xz}, \hat{\psi}_{yz}, -\nabla_1^2 \hat{\psi}\}$$

where  $\nabla_1^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$  is the two-dimensional Laplacian.

[Busse and Joseph, 1972]. Introducing

$$\hat{\psi}, \hat{s} = \{\psi(z), s(z)\} \exp(i(kx + ly) + \sigma t) \quad (13)$$

where  $k$  and  $l$  are the real wave numbers in the  $x$ - and  $y$ -direction, respectively, and  $\sigma = \sigma^r + i\sigma^i$  is the complex growth rate, we obtain from (2) that

$$(D^2 - \alpha^2)\psi = Ra s \quad (14)$$

where  $\alpha^2 = k^2 + l^2$  is the horizontal overall wave number and  $D \equiv d/dz$ .

Inserting (14) into (4), and neglecting non-linear terms, we obtain on the basis of (6), valid for small Peclet numbers, that

$$\begin{aligned} & \{(D^2 - \alpha^2)^2 - \alpha^2 Ra - \sigma(D^2 - \alpha^2) - ikU[D^2 - \alpha^2 - \epsilon_1 \alpha^2 Ra - \epsilon_2 Ra(2D^2 - \alpha^2)] \\ & - U^2(D^2 - \alpha^2)[\epsilon_1 k^2 - \epsilon_2(D^2 - l^2)]\} \psi = 0. \end{aligned} \quad (15)$$

The requirement of impermeable, perfectly conducting boundaries leads to

$$\psi = D^2\psi = 0 \quad \text{at} \quad z = \pm \frac{1}{2}. \quad (16)$$

From this, and (15), it is easily seen that all even derivatives of  $\psi$  will vanish on the boundaries. Hence the solutions must be

of the form

$$\psi \sim \begin{cases} \cos(2n-1)\pi z \\ \sin 2n\pi z \end{cases}, \quad n = 1, 2, 3 \dots \quad (17)$$

[Chandrasekhar, 1961, p.35].

The transition from stable to unstable solutions goes through  $\sigma^r = 0$  (marginal stability). Taking  $\psi \sim \cos \pi z$ , which is easily seen to be the most unstable mode, the real and imaginary parts of (15) reduce, respectively, to

$$Ra = \frac{(\pi^2 + \alpha^2)^2}{\alpha^2} + \frac{(\pi^2 + \alpha^2)}{\alpha^2} \left[ k^2 \epsilon_1 + (\pi^2 + 1^2) \epsilon_2 \right] U^2 \quad (18)$$

and

$$\sigma^i = k \left[ -1 - \frac{\alpha^2 Ra \epsilon_1}{\pi^2 + \alpha^2} + \frac{(2\pi^2 + \alpha^2) Ra \epsilon_2}{\pi^2 + \alpha^2} \right] U \quad (19)$$

For a disturbance with  $k \neq 0$ , the ratio  $-\sigma^i/k$  represents the phase speed.

The preferred mode of disturbance will make  $Ra$  a minimum, denoted as the critical Rayleigh number,  $R_c$ . Minimizing  $Ra$  with respect to  $k$  and  $l$ , and utilizing the fact that  $\epsilon_1 > \epsilon_2$ , we obtain

$$R_c = 4\pi^2(1 + \epsilon_2 U^2) \quad (20)$$

for the wave numbers

$$k_c = 0, \quad l_c = \pi \quad (21)$$

The disturbance (21) defines a longitudinal roll with axis aligned in the direction of the basic flow. The dimensionless wave length is given by  $\lambda_c = 2\pi/l_c = 2$ . From (19) we observe that longitudinal



rolls are stationary, i.e. they do not propagate relative to the basic flow.

It is seen from (20) that the critical Rayleigh number is increased due to lateral dispersion. This result is to be expected, since lateral dispersion leads to increased vertical mixing, and hence opposes the destabilizing effect of the imposed concentration gradient. Another interesting observation to make from (20) is that the breakdown of stability is independent of longitudinal dispersion.

#### Results for large Peclet numbers.

A stability analysis of the basic state (10) may also be performed in the limit of large Peclet numbers. We then assume a dispersion tensor of the form (8). Repeating the former procedure, we finally obtain at the marginal state that

$$Ra = \frac{(\pi^2 + \alpha^2)}{\alpha^2} \left[ k^2 \eta_1 + (\pi^2 + l^2) \eta_2 \right] U \quad (22)$$

and

$$\sigma^1 = k \left[ -U - \frac{\alpha^2 Ra \eta_1}{\pi^2 + \alpha^2} + Ra \eta_2 \right] \quad (23)$$

Due to the fact that  $\eta_1 > \eta_2$ , we find, as before, that stationary longitudinal rolls defined by  $k_c = 0$ ,  $l_c = \pi$  constitute the preferred mode of disturbance. This leads to a critical Rayleigh number

$$R_c = 4\pi^2 \eta_2 U . \quad (24)$$

In the derivation of (8), it is implied that  $\eta_1 U, \eta_2 U \gg 1$ . Hence we conclude from (24) that lateral dispersion very effectively delays the onset of convection in a flow with large Peclet numbers.

This latter problem has recently been analyzed by Rubin [1974]. Unfortunately he considers only two-dimensional disturbances with axes normal to the basic flow ( $k \neq 0$  and  $l = 0$  in our notation). Hence he disregards the most unstable mode, which accordingly invalidates his conclusions about marginal stability.

#### THE STABILITY OF THERMALLY-DRIVEN FLOW

In this section we consider the stability of a shear flow in a porous horizontal layer. This kind of flow may be caused by lateral temperature variations [Weber, 1974]. Let the dimensional temperature vary as  $T_* = -\beta_* x_*$  along the upper and lower boundaries, where  $\beta_*$  is a constant. The concentration difference between the boundaries is  $\Delta S$ , as before, where the highest concentration is at the upper plane. A particular solution of the dimensionless equations (2)-(5) may now be obtained by assuming

$$\begin{aligned}\partial/\partial t = v = w &= 0 \\ u &= U(z) \\ S &= S(z) \\ T &= T(z) - \beta x\end{aligned}\tag{25}$$

subject to the boundary conditions

$$S(\pm \frac{1}{2}) = \pm \frac{1}{2}, \quad T(\pm \frac{1}{2}) = 0\tag{26}$$

and to the continuity condition

$$\int_{-\frac{1}{2}}^{+\frac{1}{2}} U(z) dz = 0\tag{27}$$

By the aid of (1), the dimensionless gradient in (25) is given by

$$\beta = \frac{Kg\gamma_1 \beta_* h^2}{\nu \kappa_{mS}} \quad (28)$$

For the velocity and the temperature it is readily derived that

$$U(z) = \beta z \quad (29)$$

and

$$T(z) = \frac{1}{6} \tau \beta^2 \left( \frac{1}{4} z - z^3 \right) \quad (30)$$

In the following analysis we assume that the Peclet number is small. Substituting the velocity distribution (29) into (6), we finally obtain from (4) that

$$S(z) = \frac{\text{Arctan}(\epsilon_2^{\frac{1}{2}} \beta z)}{2 \text{Arctan}(\frac{1}{2} \epsilon_2^{\frac{1}{2}} \beta)} \quad (31)$$

where the boundary conditions (26) have been utilized. In this problem  $\epsilon_2$  is very small, and even for moderate values of  $\beta$ ,  $\epsilon_2 \beta^2 \ll 1$ . For such cases a series expansion leads to

$$S(z) = z + \frac{1}{3} \epsilon_2 \beta^2 \left( \frac{1}{4} z - z^3 \right) \quad (32)$$

valid to  $O(\epsilon^2 \beta^4)$ .

We observe from (31) or (32) that the concentration gradient has a maximum in the middle of the layer. This is to be expected since the dispersion effect now decreases inwards from the boundaries, and hence diffusion must increase in order to provide the mass balance. Perturbating the basic solutions (29), (30) and (32) by disturbances of the form (13), and utilizing (12), we derive from (2) that

$$(D^2 - \alpha^2) \psi = Ra s - \theta \quad (33)$$

where  $\theta$  is the  $z$ -dependent part of the perturbation temperature. Linearizing the solute and temperature equations(4) and (5), we finally obtain in the limit of small Peclet numbers that

$$(D^2 - \alpha^2)s - \alpha^2\psi = \sigma s - ik\beta\{-zs + 2\epsilon_2(D\psi + zD^2\psi) + (\epsilon_1 - \epsilon_2)\alpha^2z\psi\} \\ - \beta^2\{-\epsilon_2\alpha^2D\bar{S}\psi + \epsilon_2z^2(D^2 - 1^2)s + 2\epsilon_2zDs - \epsilon_1k^2z^2s\} \quad (34)$$

$$- ik\beta^3\{2\epsilon_2^2[zD^2\bar{S}D\psi + D\bar{S}(D\psi + zD^2\psi)] + \epsilon_2(\epsilon_1 - \epsilon_2)\alpha^2zD\bar{S}\psi\}$$

$$(D^2 - \alpha^2)\theta = \tau c \sigma \theta + ik\tau\beta[z\theta - D\psi] + \alpha^2\tau^2\beta^2D\bar{\theta}\psi \quad (35)$$

where we have defined

$$\bar{S} \equiv (S(z) - z)/\epsilon_2\beta^2 = \frac{1}{3}(\frac{1}{4}z - z^3) \quad (36)$$

and

$$\bar{\theta} = T(z)/\tau\beta^2 = \frac{1}{6}(\frac{1}{4}z - z^3)$$

The equations (33)-(35) are subject to the boundary conditions

$$\psi = s = \theta = 0 \quad \text{at} \quad z = \pm \frac{1}{2} \quad (37)$$

The system of coupled ordinary differential equations stated above can only be solved approximately. In the present paper we assume that the horizontal temperature variation  $\beta$  is small, and hence the solutions may be written as series expansions in  $\beta$ . Accordingly we introduce

$$\psi, s, \theta, Ra, \sigma, k, l = \sum_{n=0}^{\infty} \beta^n (\psi_n, s_n, \theta_n, R_n, \sigma_n, k_n, l_n) \quad (38)$$

where the velocity, solute and temperature fields are required to

satisfy the boundary conditions at each order.

By substituting these expansions into (33)-(35) and equating equal powers of  $\beta$ , an infinite set of inhomogeneous differential equations is obtained.  $R_1, R_2, R_3, \dots$  are found from the solvability conditions for these equations, and the wave number terms  $k_0, l_0, k_1, l_1, \dots$  are determined so that they minimize the Rayleigh number.

As in a similar problem [Weber, 1974], physical considerations give as a preliminary (and simplifying) result that the Rayleigh and wave numbers should not contain odd powers of  $\beta$ .

To order  $\beta^0$ , we observe from (35) that the perturbation temperature decays exponentially in time. This is obvious, since we only have diffusion of heat to this order. Hence we take  $\theta_0 \equiv 0$ .

The zeroth-order system then corresponds to ordinary convection without shear, and is easily shown to be self-adjoint. Hence marginal stability is given by  $\sigma_0 = 0$ , and the equations are

$$(D^2 - \alpha_0^2)\psi_0 - R_0 s_0 = 0 \tag{39}$$

$$(D^2 - \alpha_0^2)s_0 - \alpha_0^2 \psi_0 = 0$$

Eliminating  $s_0$ , we obtain

$$L\psi_0 \equiv \{(D^2 - \alpha_0^2)^2 - \alpha_0^2 R_0\}\psi_0 = 0 \tag{40}$$

subject to  $\psi_0 = D^2\psi_0 = 0$  at  $z = \pm \frac{1}{2}$ . The most unstable solution is given by

$$\psi_0 = A \cos \pi z \tag{41}$$

where  $A$  is a constant.

This yields a critical Rayleigh number

$$R_0 = 4\pi^2 \quad \text{for } \alpha_0^2 = k_0^2 + l_0^2 = \pi^2 \quad (42)$$

as originally derived by Lapwood [1948].

From (39) we obtain

$$s_0 = -\frac{A}{2} \cos \pi z \quad (43)$$

Since the zeroth-order system is self-adjoint, the condition for the higher order equations to have a non-trivial solution may be stated as

$$\int_{-\frac{1}{2}}^{+\frac{1}{2}} \psi_0 L \psi_n dz = 0, \quad n = 1, 2, 3, \dots \quad (44)$$

where the operator  $L$  is defined by (40).

The higher order equations are readily obtained, and their solution is an elementary but lengthy task. To save space, we have placed them in the appendix. The computations have not been carried further than necessary to obtain the first correction on the critical Rayleigh number, that is to  $O(\beta^2)$ .

Applying the solvability condition (44) to the first- and second-order system (A1) and (A2), it immediately follows that

$$\sigma_1^1 = \sigma_2^1 = 0 \quad (45)$$

at the marginal state. Hence we have no oscillatory behaviour to second order.

From the application of (44) to (A2), we further obtain

$$R_2 = \frac{\tau^2}{4\pi^2} + a\pi^2 \epsilon_2 + (a\epsilon_1 + b)k_0^2 + a\epsilon_2 l_0^2 \quad (46)$$

where  $a = 1/6 - 1/\pi^2$ , and  $b$  is a positive constant which is not stated explicitly for the sake of simplicity. From (46) we observe that  $R_2$  has a minimum for  $k_0 = 0$ , and hence  $l_0 = \pi$ , i.e. a longitudinal roll.

To second order then, stationary longitudinal rolls with dimensional wavelength  $2h$  constitute the preferred mode of disturbance. The critical Rayleigh number is given by

$$R_c = 4\pi^2 + \left\{ \frac{\tau^2}{4\pi^2} + \left( \frac{\pi^2}{3} - 2 \right) \epsilon_2 \right\} \beta^2 \quad (47)$$

The first term in the parenthesis on the right of (47) results from the stabilizing temperature configuration (30) in the basic flow, and conforms to the findings of previous studies [Weber, 1973, 1974]. The second term in the parenthesis exhibits, as in our former investigations, the stabilizing effect of lateral dispersion.

In this connection it might be interesting to look back on the basic concentration distribution (32). It is seen that the gradient exceeds unity in the middle by the amount  $\epsilon_2 \beta^2 / 12$ . Since this means a possibility for exchange of energy from the basic flow to the perturbations, one would expect this effect to support instability. This is indeed confirmed by the computations. However, one also finds that the stabilizing effect of vertical mixing due to lateral dispersion of perturbation solute dominates, at least to this order. Hence, in total, dispersion acts stabilizing as concluded before.

## SUMMARY AND CONCLUDING REMARKS

The effect of hydrodynamic dispersion on buoyancy-driven convection in stratified flows through porous media has been studied. Both a pressure-driven uniform flow and a thermally-driven shear flow have been considered. The stability analysis has been performed on the basis of linear theory, and the obtained results are valid for isotropic media in the limit of small and large Peclet numbers. The main results, being essentially similar for all cases, can be summarized as follows :

- (i) Lateral dispersion always leads to an increase of the critical Rayleigh number when compared to ordinary convection without a basic flow.
- (ii) The onset of convection is independent of the longitudinal dispersion coefficient.
- (iii) The preferred mode of instability is stationary two-dimensional disturbances having axis oriented along the basic flow (longitudinal rolls).

Before closing, we emphasize that the results in the last section have been obtained under the assumption that  $\beta$  is a small parameter. It may also be worth mentioning that, for sufficiently large values of  $\beta$ , the temperature gradient in the basic state could be large enough for double-diffusive phenomena to occur; see the review by Turner [1974]. Since the region in the middle is characterized by hot salty water above cold fresh water, a tendency



towards a salt finger type of flow would be expected. In this case an analytical treatment is complicated by the fact that series expansions of the type (38) do not converge. Accordingly a numerical approach should be considered.

### APPENDIX

By substituting (38) into (33)-(35) we derive the following set of equations :

$$O(\beta^1): (D^2 - \alpha_0^2)\psi_1 - R_0 s_1 = -\theta_1$$

$$(D^2 - \alpha_0^2)s_1 - \alpha_0^2\psi_1 = \sigma_1 s_0 - ik_0\{-zs_0 + 2\epsilon_2(D\psi_0 + zD^2\psi_0) + (\epsilon_1 - \epsilon_2)\alpha_0^2 z\psi_0\} \quad (A1)$$

$$(D^2 - \alpha_0^2)\theta_1 = -ik_0\tau D\psi_0$$

$$O(\beta^2): (D^2 - \alpha_0^2)\psi_2 - R_0 s_2 = \alpha_2^2\psi_0 + R_2 s_0 - \theta_2$$

$$(D^2 - \alpha_0^2)s_2 - \alpha_0^2\psi_2 = \alpha_2^2 s_0 + \alpha_2^2\psi_0 + \sigma_2 s_0 - ik_0\{-zs_1 + 2\epsilon_2(D\psi_1 + zD^2\psi_1) + (\epsilon_1 - \epsilon_2)\alpha_0^2 z\psi_1\} \quad (A2)$$

$$-\{ -\alpha_0^2\epsilon_2 D\bar{S}\psi_0 + \epsilon_2 z^2(D^2 - l_0^2)s_0 + 2\epsilon_2 zDs_0 - \epsilon_1 k_0^2 z^2 s_0 \}$$

$$(D^2 - \alpha_0^2)\theta_2 = ik_0\tau(z\theta_1 - D\psi_1) + \alpha_0^2\tau^2 D\bar{\Theta}\psi_0$$

Here  $R_0 = 4\pi^2$ ,  $\alpha_0^2 = k_0^2 + l_0^2 = \pi^2$  and  $\alpha_2^2 = 2(k_2 k_0 + l_2 l_0)$ .

Applying the normalization condition

$$\int_{-\frac{1}{2}}^{+\frac{1}{2}} \psi_0 \psi dz = \frac{1}{2} \quad (A3)$$

which makes the constant in (41) equal to unity, we obtain from (A1) that

$$\psi_1 = -\frac{ik_0}{2\pi} \{a_1 \sin \pi z + a_2 \sinh(\pi\sqrt{3}z) + a_3 z \cos \pi z + a_4 z^2 \sin \pi z\} \quad (A4)$$

$$\theta_1 = \frac{ik_0\tau}{2\pi} \{-\sin \pi z + \frac{\sinh(\pi z)}{\sinh(\pi/2)}\} \quad (A5)$$

while  $s_1$  is found from the equation

$$s_1 = \frac{1}{4\pi^2} \{(D^2 - \pi^2)\psi_1 + \theta_1\} \quad (A6)$$

In (A4) the coefficients are

$$\begin{aligned} a_1 &= \frac{1}{4} \left\{ \frac{1}{2\pi^2} (1-\tau) + \frac{1}{4} + \left(1 + \frac{\pi^2}{2}\right) \epsilon_1 - \left(\frac{3\pi^2}{2} - 1\right) \epsilon_2 \right\} \\ a_2 &= -\frac{1}{4\pi \sinh(\pi\sqrt{3}/2)} \left\{ \frac{1}{2\pi} (1-\tau) + \pi \epsilon_1 + \pi \epsilon_2 \right\} \\ a_3 &= -\left\{ \frac{1}{2\pi} \left(1 - \frac{\tau}{2}\right) + \pi \epsilon_1 - \pi \epsilon_2 \right\} \\ a_4 &= -\frac{1}{2} \left\{ \frac{1}{2} + \pi^2 \epsilon_1 - 3\pi^2 \epsilon_2 \right\} \end{aligned} \quad (A7)$$

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